HERCULES AND THE HYDRA: A PROBLEM IN COMBINATORICS

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Abstract

A tree $T=(V, E)$ is a connected graph which is either void, or contains a leaf, i.e. a vertex v with at most one incident edge e, and the subgraph $T'=(V-\{v\}, E-\{e\})$ still is a tree. This is one of the possible definitions of a tree, and in fact is the most adequate for discussing a particular problem on trees.

Hercules and the Hydra.
The Hydra was a monster with many heads living in a marsh of Lerna. Hercules fought the Hydra chopping her heads with a club, however, as he chopped a head, two new heads grew forth from the monster's body. Hercules won; and in fact we shall see that no matter in which order he had attacked the heads, the monster would have been annihilated.

Although less charming we shall treat the Hydra as a rooted tree and her heads as leaves, specifying the rules for head chopping and re-growing. In particular, no new leaf is allowed to grow forth from the root. The problem was formulated by Kirby and Paris (*Bull. of London Math. Soc.*, 1982,) who proved the convergence to zero of the number of heads using transfinite induction, and how head chopping is related to some successions of integers. To the best of our knowledge an elementary proof of convergence is a challenging open problem.

We shall also discuss a new chopping strategy supported by an elementary proof of convergence, an extension of the problem to graphs, and its possible applications in distributed systems.